Cauchy-Schwarz and Other Classical Inequalities

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1 Introduction

Welcome to my inequalities talk! As always, remember that these notes cover only a fraction of the subject, and you should find more to read if you are serious about doing well at the IMO. In the case of inequalities, there are many extensive notes available online:

- "Topics in Inequalities - Theorems and Techniques" by Hojoo Lee
- "Olympiad Inequalities" by Thomas J. Mildorf
- "Basics of Olympiad Inequalities" by Samin Riasat

Just search for them on Google. There are some Canadian notes available here:

http://sites.google.com/site/imocanada/.

Let me also add a short word on notation. I will frequently write things like $\sum_{\text{cyc}} a^2b$ and $\sum_{\text{sym}} a^2b$. These are best explained by example. For three variables a, b, c:

$$\sum_{\text{cyc}} a^2b = a^2b + b^2c + c^2a, \qquad \text{and} \qquad \sum_{\text{sym}} a^2b = a^2b + b^2a + b^2c + c^2b + c^2a + a^2c,$$

$$\sum_{\text{cyc}} abc = abc + bca + cab = 3abc, \qquad \text{and} \qquad \sum_{\text{sym}} abc = abc + bac + bca + cba + cab + acb = 6abc,$$

$$\sum_{\text{cyc}} a^3 = a^3 + b^3 + c^3, \qquad \text{and} \qquad \sum_{\text{sym}} a^3 = a^3 + a^3 + b^3 + b^3 + c^3 + c^3.$$

If there are n variables, $\sum_{\text{cyc}}$ will always have n terms, and you get from one term to the others by shifting which variable comes first. The relative order stays the same though. On the other hand, $\sum_{\text{sym}}$ will always have $n!$ terms, and you get from one term to the others by trying all $n!$ re-orderings of the variables (even if not all n variables appear in each term).

This notation – called cyclic and symmetric sums – is a very convenient short-hand. Be sure to ask if you are confused!

2 The Tool Chest

The first step to solving inequality problems is learning the relevant theorems. Probably you have seen most of these before, but I will list them here for completeness.

2.1 Try These First!

AM-GM-HM:

For $x_1, x_2, \ldots, x_n \geq 0$,

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}.$$

Equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Cauchy-Schwarz:

For $x_1, x_2, \ldots, x_n \geq 0$ and $y_1, y_2, \ldots, y_n \geq 0$:

$$\left(\sum_{i=1}^{n} x_i \cdot y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right).$$

On contests, you will often want to use one of the two following forms instead:

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} y_i}$$
$$\sum_{i=1}^{n} \frac{x_i}{y_i} = \sum_{i=1}^{n} \frac{x_i^2}{x_i y_i} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} x_i y_i}.$$

The first of these two forms is sometimes called "Engel" form. In all cases, equality holds if and only if $\frac{x_i}{y_i}$ is the same for all i.

2.2 Plan B

"Hölder":

For $x_{i,j} \geq 0$ with $1 \leq i \leq m$ and $1 \leq j \leq n$:

$$\left(\sum_{i=1}^{n} x_{i,1} \cdot x_{i,2} \cdot \ldots \cdot x_{i,m}\right)^m \leq \left(\sum_{i=1}^{n} x_{i,1}^m\right) \cdot \left(\sum_{i=1}^{n} x_{i,2}^m\right) \cdot \ldots \cdot \left(\sum_{i=1}^{n} x_{i,m}^m\right).$$

This inequality is a natural generalization of Cauchy-Schwarz, making it easy to remember and sometimes easy to use. The biggest trick is figuring out what to call it. Some people say Hölder's, but not everyone will recognize it by that name[1]. If you want to use it on a contest, you should

[1] Traditionally, Hölder's inequality is stated as a weighted version of Cauchy-Schwarz: For any positive numbers $(x_i)_{i=1}^n, (y_i)_{i=1}^n, p, q$ with $p + q = 1$, it is true that $\sum_{i=1}^n x_i^p y_i^q \leq \left(\sum_{i=1}^n x_i\right)^p \cdot \left(\sum_{i=1}^n y_i\right)^q$. Do you see why these two formulations are equivalent?

state the full result first. That way there will be no confusion.

Rearrangement:

Suppose $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. If σ is an arbitrary permutation of $\{1, 2, \ldots, n\}$. then:

$$\sum_{i=1}^{n} x_i \cdot y_i \geq \sum_{i=1}^{n} x_i \cdot y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i \cdot y_{n+1-i}.$$

The main advantage of the rearrangement inequality is that it works even for negative numbers, something that is not true for most of the other theorems. Bear that in mind!

Power-Mean:

Define the order-p power mean of $x_1, x_2, \ldots, x_n$ to be:

$$\begin{aligned} \left(\frac{x_1^p + x_2^p + \ldots + x_n^p}{n}\right)^{1/p} & \qquad \text{if } p \neq 0 \\ \sqrt[n]{x_1 \cdot x_2 \cdot \ldots x_n} & \qquad \text{if } p = 0. \end{aligned}$$

If $x_1, x_2, \ldots, x_n \geq 0$ and $p \geq q$, then the order-p power mean of $x_1, x_2, \ldots, x_n$ is at least as large as the order-q power mean.

Jensen:

A continuous function $f : [a, b] \to \mathbb{R}$ is called "convex" if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$ for all x, y. If f is convex, then for all $x_1, x_2, \ldots, x_n \in [a, b]$:

$$\frac{1}{n} \cdot \sum_{i=1}^{n} f(x_i) \geq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right).$$

If $\frac{f(x)+f(y)}{2} \leq f\left(\frac{x+y}{2}\right)$ for all x, y, then f is called "concave" and the opposite of this inequality is true.

Jensen's is most useful for people who know calculus, because you can prove a function $f(x)$ is convex by calculating its second derivative and proving $f''(x) > 0$ for all x. Alternatively, you can just memorize some convex functions:

- x^r is convex for $x \geq 0$ and $r \notin (0, 1)$.
- $\sin x$ is concave for $x \in [0, \pi]$, while $\cos x$ is concave for $x \in [-\pi/2, \pi/2]$.
- $\tan x$ is convex for $x \in [0, \pi/2)$.
- e^x is convex for all x, while $\log x$ is concave for all x.
- The sum of two convex functions is convex.

2.3 The Rest

Weighted AM-GM:

If $w_1, w_2, \ldots, w_n \geq 0$ satisfy $w_1 + w_2 + \ldots + w_n = 1$, and $x_1, x_2, \ldots, x_n \geq 0$, then:

$$w_1 \cdot x_1 + w_2 \cdot x_2 + \ldots + w_n \cdot x_n \geq x_1^{w_1} \cdot x_2^{w_2} \cdot \ldots \cdot x_n^{w_n}.$$

For rational $w_1, w_2, \ldots, w_n$, this is equivalent to just repeating some terms in the regular AM-GM inequality. Using similar tricks, you can also get a weighted version of Jensen's inequality.

Muirhead:

We say $(a_1, a_2, \ldots, a_n)$ *majorizes* $(b_1, b_2, \ldots, b_n)$ if:

- $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$,
- $a_1 + a_2 + \ldots + a_k \geq b_1 + b_2 + \ldots + b_k$ for $k = 1, 2, \ldots, n$, and
- $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n$.

If $(a_1, a_2, \ldots, a_n)$ majorizes $(b_1, b_2, \ldots, b_n)$ and $x_1, x_2, \ldots, x_n \geq 0$, then:

$$\sum_{\text{sym}} x_1^{a_1} \cdot x_2^{a_2} \cdot \ldots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} \cdot x_2^{b_2} \cdot \ldots \cdot x_n^{b_n},$$

Remember that the symmetric sum is taken over all $n!$ permutations of $x_1, x_2, \ldots, x_n$.

Muirhead is similar to coordinate geometry in many ways. Given an inequality, you can often expand it out, group terms, and then apply Muirhead without ever thinking. Just don't make a mistake! The IMO jury is well aware of Muirhead however, and recent problems are not easily solved in this way.

Schur:

If $x, y, z \geq 0$ and $r > 0$, then:

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

If $r = 1$, this simplifies to:

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2x + y^2z + z^2y + z^2x + x^2z.$$

Equality holds if $x = y = z$, or if two values are equal and the third is zero.

Schur's inequality is not considered standard knowledge, and so you will never *need* it to solve a problem. However, it is an important tool if you decide to multiply an inequality out. It is one of

very few inequalities that can prove xyz is *greater* than something else.

3 The Many Faces of Cauchy-Schwarz and Hölder

Back in the good old days, inequality problems were pretty straightforward. Pick one of the classic theorems, apply it in the obvious way, and often that would be enough by itself. Here's a pretty typical example:

Example 1 (APMO 1991, #3)**.** *Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n$. Show that*

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \ldots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1 + a_2 + \ldots + a_n}{2}.$$

Solution. By Cauchy-Schwarz in Engel form:

$$\begin{aligned} \sum_{i=1}^n \frac{a_i^2}{a_i+b_i} &\geq \frac{\left(\sum_{i=1}^n a_i\right)^2}{\sum_{i=1}^n (a_i+b_i)} \\ &= \frac{\left(\sum_{i=1}^n a_i\right)^2}{2 \cdot \sum_{i=1}^n a_i} \\ &= \frac{\sum_{i=1}^n a_i}{2}. \end{aligned}$$

□

These days however, inequalities are always disguised, and you need to combine the theorems with some other kinds of manipulation. In this section, I will go over a few useful tricks that come up again and again. To emphasize the tricks rather than the theorems, each of the main examples will ultimately come down to Cauchy-Schwarz or Hölder.

More Examples:

1. Prove that $(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$ for $a, b, c \geq 0$.
2. Let $P(x)$ be a polynomial with positive coefficients.
   Prove that if $P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$ holds for $x = 1$, then it holds for all $x > 0$.
3. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^n x_i = \sum_{i=1}^n \frac{1}{x_i^2}$.
   Prove that $\sum_{i=1}^n x_i \geq n$.

3.1 Move Things Around First

One of the simplest ways to disguise an inequality is to move some terms around so that the equation is no longer in exactly the right form. Here's an example:

Example 2. *Prove that $\frac{b}{a+2b+c} + \frac{c}{b+2c+d} + \frac{d}{c+2d+a} + \frac{a}{d+2a+b} \leq 1$ for $a, b, c, d \geq 0$.*

Any time we are trying to bound a sum of fractions, Cauchy-Schwarz is the first inequality to try. However, in both fractional forms (see Page 1), we really want to show the fractions are *greater* than something, not less than something. So we're screwed here, right? Nope! Instead of proving $\frac{b}{a+2b+c}$ is small, we could just prove $\frac{1}{2} - \frac{b}{a+2b+c} = \frac{0.5a}{a+2b+c} + \frac{0.5c}{a+2b+c}$ is big! And once you phrase the problem like that, Cauchy-Schwarz works just fine.

Solution. By Cauchy-Schwarz,

$$\begin{aligned}\sum_{\text{cyc}} \frac{a}{a+2b+c} &\geq \frac{(a+b+c+d)^2}{\sum_{\text{cyc}}(a^2+2ab+ac)} \\ &= \frac{(a+b+c+d)^2}{\sum_{\text{cyc}} a^2 + 2\sum_{\text{cyc}} ab + 2ac + 2bd} = 1.\end{aligned}$$

Similarly, $\sum_{\text{cyc}} \frac{c}{a+2b+c} \geq 1$. Therefore, $\sum_{\text{cyc}} \frac{b}{a+2b+c} = 2 - \sum_{\text{cyc}} \left(\frac{0.5a}{a+2b+c} + \frac{0.5c}{a+2b+c}\right) \leq 1$. □

Notice that we could have instead tried to use Cauchy's inequality in Engel form. This states that $\sum_{\text{cyc}} \frac{a}{a+2b+c} \geq \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)^2}{4(a+b+c+d)}$. But now what happens if you plug in some random numbers, like $a = b = c = 0$ and $d = 1$? Oops! The right-hand side isn't bigger than 1 at all, which means you're screwed and you have to start over.

This gives two bonus lessons:

- After applying some theorems, try plugging in values to the thing you want to prove. If it isn't even true, then you're back to square one!
- Cauchy-Schwarz is very sensitive to the slightest tweak. The difference between the correct approach and the wrong approach to bounding $\sum_{\text{cyc}} \frac{a}{a+2b+c}$ is whether you apply Cauchy-Schwarz in Engel form directly, or whether you first write it as $\sum_{\text{cyc}} \frac{a^2}{a^2+2ab+ac}$, and then apply Cauchy-Schwarz. (That is where the third form comes from.)

More Examples:[2]

1. If $a, b, c > 0$, prove that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2+9}{a^2+b^2+c^2+3}$.
2. Let a, b, c be positive real numbers satisfying $a^2 + b^2 + c^2 = 3$.
   Prove that $\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1$.
3. Let $x_1, x_2, \ldots, x_n > 0$.
   Prove that $\frac{x_1^3}{x_1^2+x_1x_2+x_2^2} + \frac{x_2^3}{x_2^2+x_2x_3+x_3^2} + \ldots + \frac{x_n^3}{x_n^2+x_nx_1+x_1^2} \geq \frac{1}{3}(x_1 + x_2 + \ldots + x_n)$.
4. Let a, b, c be the sides of a triangle.
   Prove that $\frac{b^2+c^2-a^2}{(b+c-a)^2} + \frac{c^2+a^2-b^2}{(c+a-b)^2} + \frac{a^2+b^2-c^2}{(a+b-c)^2} \leq 3$.

3.2 Think About Equality Cases

Consider the following problem:

Example 3 (Russia)**.** *If $a, b, c, d \geq 0$, prove that*

$$\sqrt[3]{ab} + \sqrt[3]{cd} \leq \sqrt[3]{a+c+d} \cdot \sqrt[3]{a+c+b}.$$

[2]The "More Examples" in these notes can be quite tricky and use the key idea in new and creative ways. Think about them for 5-10 minutes before giving up, but don't be afraid to look at the hints if you get stuck! Unlike the main examples, these extra examples do not necessarily use Cauchy-Schwarz or Hölder.

If you cube everything, this looks pretty close to Hölder's inequality. But there are a couple problems: (1) there are only two summands on the left-hand side while each factor on the right-hand side has three summands, and (2) to get cube roots coming out, there should be three factors on the right-hand side instead of two.

Without any kind of additional guidance, you might first try grouping up factors on the right-hand side like this:

$$\sqrt[3]{a+c+d} \cdot \sqrt[3]{b+a+c} \cdot \sqrt[3]{1+1+1} \geq \sqrt[3]{ab} + \sqrt[3]{ac} + \sqrt[3]{cd}.$$

This looks pretty good, right? It is a direct application of Hölder and it looks pretty close to what we are trying to prove. But actually this deduction can't possibly help. Here's why: equality holds in this step only if $\frac{a}{b} = \frac{d}{c}$. Notice that this does not include $a = c = 1$ and $b = d = 2$, but equality *does* hold in that case for the original problem. So when we need our approximation to be exact, this application of Hölder is not exact, and therefore we're screwed.

The good news is the equality case can help point us in a better direction. In particular, we know that b and d can actually equal $a + c$. This suggests that maybe we should think of $a + c$ as a single summand when we are grouping with b and d. Once you get that idea, the rest of the problem falls apart:

Solution. By Hölder's inequality:

$$\left(\sqrt[3]{ab(a+c)} + \sqrt[3]{c(a+c)d}\right)^3 \leq (a+c) \cdot \Big(b + (a+c)\Big) \cdot \Big((a+c) + d\Big).$$

Dividing through by $(a + c)$ and taking cube roots gives us the desired result. □

The moral here is you should always, always, always try to figure out what the equality cases are in an inequality. They will usually be something simple like $x = y = z$, but they won't always be. And then when you apply any theorems, make sure equality in the theorem holds when it does in the problem. Otherwise, the theorem is too weak and cannot possibly prove what you need!

More Examples:

1. For all positive reals a, b, c, d, show that $\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}$
2. Let x, y, z be positive real numbers such that $xyz = 1$.
   Prove that $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$.
3. Let x, y, z be three real numbers, all different from 1, such that $xyz = 1$.
   Prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$.

3.3 Substitute

Next, let's take a look at an example that shamelessly rips off #2 from IMO 1995:

Example 4 (USA TST 2010)**.** *Let a, b, c be positive real numbers satisfying $abc = 1$. Show that*

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

This looks pretty intimidating, and a direct attack is unlikely to get you anywhere. However, it magically simplifies to something much more tractable with a change of variables:

Solution. Let $x = \frac{1}{a}, y = \frac{1}{b}$, and $z = \frac{1}{c}$. Then $xyz = \frac{1}{abc} = 1$ and $\frac{1}{a^5(b+2c)^2} = \frac{x^5y^2z^2}{(z+2y)^2} = \frac{x^3}{(z+2y)^2}$. Therefore, we need to show that if $xyz = 1$, then $\sum_{\text{cyc}} \frac{x^3}{(z+2y)^2} \geq \frac{1}{3}$. Indeed, Hölder's inequality guarantees that:

$$\left(\sum_{\text{cyc}} \frac{x^3}{(z+2y)^2}\right) \cdot \left(\sum_{\text{cyc}} (z+2y)\right) \cdot \left(\sum_{\text{cyc}} (z+2y)\right) \geq (x+y+z)^3$$

$\sum_{\text{cyc}}(z+2y) = 3(x+y+z)$, so we can divide through by $9(x+y+z)^2$ to get:

$$\sum_{\text{cyc}} \frac{x^3}{(z+2y)^2} \geq \frac{x+y+z}{9} \geq \frac{\sqrt[3]{xyz}}{3} = \frac{1}{3}.$$ □

Here are some other substitutions that are useful:

- a, b, c are the sides of a triangle if and only if the following variables are positive: $x = \frac{b+c-a}{2}, y = \frac{c+a-b}{2}, z = \frac{a+b-c}{2}$. Try using x, y, z instead of a, b, c on any problem that involves the side lengths of a triangle. This is called the "Ravi substitution" after Canadian IMO veteran Ravi Vakil.
- You have already seen what $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$ can do. It can be a little tricky to predict when this substitution will help, so just try it if you are stuck.
- If $abc = 1$, you can let $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$. This will sometimes turn an asymmetric inequality into something much nicer, but it can definitely make things worse too!
- More exotically, you can also do trigonometry substitutions. If $a, b, c > 0$, then:
  - $a + b + c = abc$ is equivalent to saying $a = \tan X, b = \tan Y, c = \tan Z$ for X, Y, Z the angles of an acute triangle.
  - $ab + bc + ca = 1$ is equivalent to saying $a = \cot X, b = \cot Y, c = \cot Z$ for X, Y, Z the angles of an acute triangle.
  - $a^2 + b^2 + c^2 + 2abc = 1$ is equivalent to saying $a = \cos X, b = \cos Y, c = \cos Z$ for X, Y, Z the angles of an acute triangle.

  Furthermore $\sqrt{\frac{1}{a^2+1}} = \cos X$ if $a = \tan X$, and $\sqrt{1-a^2} = \sin X$ if $a = \cos X$. If any of these statements would simplify a problem, consider making the corresponding trig substitution!

More Examples:

1. Let a, b, c be positive real numbers satisfying $abc = 1$.
   Prove that $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$.
2. Let a, b, c be positive real numbers satisfying $abc = 1$.
   Prove that $\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$.
3. Let a, b, c be non-negative real numbers satisfying $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$.
   Prove that $ab + bc + ca \leq \frac{3}{2}$.

3.4 Isolated Fudging

Cauchy-Schwarz gives you the ability to bound $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}$ all at once without looking at each term too carefully. Very few other theorems give you that power. When you can't look at the whole sum as one quantity, it is useful to prove each individual term is bigger (or smaller) than something much easier to work with. Usually, you will want the fudged terms to all have the same denominator, so that you can easily add them up. Here is an example:

Example 5 (IMO 2005, #3)**.** *Let x, y, and z be positive numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

Solution. The given inequality is equivalent to:

$$\sum_{\text{cyc}}\left(\frac{x^2-x^5}{x^5+y^2+z^2}+1\right) \leq 3 \qquad \Longleftrightarrow \qquad \sum_{\text{cyc}}\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq 3.$$

Now, Cauchy-Schwarz implies that $(x^5+y^2+z^2)(yz+y^2+z^2) \geq (\sqrt{x^5yz}+y^2+z^2)^2 \geq (x^2+y^2+z^2)^2$. Therefore,

$$\begin{aligned}\sum_{\text{cyc}}\frac{x^2+y^2+z^2}{x^5+y^2+z^2} &\leq \sum_{\text{cyc}}\frac{yz+y^2+z^2}{x^2+y^2+z^2}\\ &= \frac{2x^2+2y^2+2z^2+xy+yz+zx}{x^2+y^2+z^2} \leq 3\end{aligned}$$

□

This is a difficult problem, but once you write the inequality as $\sum_{\text{cyc}}\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq 3$, Cauchy-Schwarz makes it very natural to compare each term with $\frac{\text{something}}{x^2+y^2+z^2}$. And once you get that far, you should be able to figure out what the "something" has to be.

This was an example where the fudged terms were suggested by just trying various simplifications on the original terms. In other cases, it can be helpful to guess that there is a fudged term in some simple form such as $\frac{a}{a+b+c}$, and then try to prove it works. Sometimes the guessing can be quite tricky though! For example, you can prove $\sum_{\text{cyc}}\frac{a}{\sqrt{a^2+8bc}} \geq \sum_{\text{cyc}}\frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} = 1$, but why would you think to try this?[3] See Yufei's notes for some tips: `http://sites.google.com/site/imocanada/2008-winter-camp`.

More Examples:

1. Let a, b, c be positive real numbers satisfying $a^2+b^2+c^2 = 3$.
   Prove $\frac{a}{a^2+b+c}+\frac{b}{b^2+c+a}+\frac{c}{c^2+a+b} \leq 3$.
2. If $a, b, c > 0$, show that $\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$.
3. If $a, b, c > 0$, show that $\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}+\left(\frac{2b}{c+a}\right)^{\frac{2}{3}}+\left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3$.
4. If $x, y, z \geq 0$ and $x^2+y^2+z^2 = 1$, show that $\sum_{\text{cyc}}\sqrt{1-xy}\cdot\sqrt{1-yz} \geq 2$.

[3] This problem comes from IMO 2001 #2. Can you see a direct proof that $\sum_{\text{cyc}}\frac{a}{\sqrt{a^2+8bc}} \geq 1$ using Hölder's?

4 Problems

I will end with a fairly random collection of Olympiad inequality problems for you to practice on and challenge yourself with. The preceding sections might help you, but they also might not. A-level problems might be on the CMO, B-level problems might be on the IMO, and C-level problems are some of the hardest inequalities ever proposed!

A1. Prove that $3a^2 + 3b^2 + c^2 \geq 2ab + 2bc + 2ca$ for any real numbers a, b, c.

A2. Prove that $\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{1997}{1998} < \frac{1}{44}$.

A3. Let a, b, c be positive real numbers for which $a + b + c = 1$.
Prove that $\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2}$.

A4. If $a, b, c > 0$, prove that $\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2 \cdot \left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$.

A5. Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.

(a) Prove that for every sequence, there is an $n > 1$ such that $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999$.

(b) Find a sequence such that for all n: $\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4$.

A6. Let a, b and c be the lengths of the sides of a triangle.
Prove that $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$.

B1. If $a, b, c > 0$, prove that $\frac{\max(a,b,c)}{\min(a,b,c)} + \frac{\min(a,b,c)}{\max(a,b,c)} \geq \frac{a+b+c}{\sqrt[3]{abc}} - 1$.

B2. Let $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$.
Prove that $\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$.

B3. If $a, b, c > 0$, prove that $\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc}$.

B4. Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for $i, j = 1, 2, \ldots$.
Prove that $a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_n}{n} \geq a_n$.

B5. If $a, b, c \geq 0$, prove that $\frac{a}{\sqrt{4b^2+bc+4c^2}} + \frac{b}{\sqrt{4c^2+ca+4a^2}} + \frac{c}{\sqrt{4a^2+ab+4b^2}} \geq 1$.

B6. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the orthogonal projections of S onto the yz, zx, xy planes, respectively. Show that $|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$.

B7. Let $n > 3$ and $a_1, a_2, \ldots, a_n$ be positive numbers satisfying $a_1 + a_2 + \ldots + a_n = 2$.
Determine the minimum value of $\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \ldots + \frac{a_n}{a_1^2+1}$.

C1. Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be non-negative real numbers.
Prove that $\sum_{i,j} \min(a_ia_j, b_ib_j) \leq \sum_{i,j} \min(a_ib_j, b_ia_j)$.

C2. Let $n \geq 2$ be a positive integer, and let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, z_3, \ldots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \geq x_iy_j$ for all $i, j \in \{1, 2, \ldots, n\}$. Let $M = \max(z_2, z_3, \ldots, z_{2n})$.
Prove that $\left(\frac{M+z_2+z_3+\ldots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\left(\frac{y_1+y_2+\ldots+y_n}{n}\right)$.

5 Hints

More Examples: The Many Faces of Cauchy-Schwarz and Hölder

1. Write $(a^3+2)(b^3+2)(c^3+2) = (a^3+1+1)\cdot(1+b^3+1)\cdot(1+1+c^3)$.
2. What happens when you apply Cauchy-Schwarz to $\left(\sum_{i=1}^n \frac{a_i}{x^i}\right)\cdot\left(\sum_{i=1}^n a_i x^i\right)$? (Source: Titu Andreescu)
3. By Hölder, $\left(\sum_{i=1}^n x_i\right)^2 \cdot \left(\sum_{i=1}^n \frac{1}{x_i^2}\right) \geq n^3$. (Source: Romania 2007)

More Examples: Move Things Around First

1. The numerator on the right-hand side is mysterious, but the denominators suggest Cauchy-Schwarz in Engel form. We just need to switch the sign around, as in the worked example. (Source: Adrian Tang)
2. AM-GM guarantees $a^3+1+1 \geq 3a$, but applying that directly only gives $\frac{1}{a^3+2} \leq \frac{1}{3a}$. (Source: Pham Kim Hung)
3. How does the left-hand side compare with $\sum_{\text{cyc}} \frac{x_1^3+x_2^3}{x_1^2+x_1x_2+x_2^2}$? Once you have the inequality in this form, you should be able to bound each term separately. (Source: ???)
4. Write the inequality as $\sum_{\text{cyc}} \frac{(a-b)(a-c)}{(b+c-a)^2} \leq 3$. We can assume without loss of generality that $a \geq b \geq c$. Which of the three summands is positive? Which has largest absolute value? (Source: Adapted from IMO Shortlist 2006, A5)

More Examples: Think About Equality Cases

1. Equality holds if $a=b=1, c=2, d=4$. Maybe you could split up some terms. (Source: ???)
2. By AM-GM, $\frac{x^3}{(1+y)(1+z)}+(1+y)+(1+z) \geq 3x$. This would make the inequality much simpler, but unfortunately, equality does not hold at the right place (namely $x=y=z$). What if you used $\frac{1+y}{C}$ instead of just $1+y$? (Source: IMO Short List 1998, A3)
3. Since $x=y=z$ is not an equality case, none of our theorems work! Let $a=\frac{x}{1-x}, b=\frac{y}{1-y}, c=\frac{z}{1-z}$, and see what the given condition turns into. (Source: IMO 2008, #2a)

More Examples: Substitute

1. This is almost identical to the worked example. Substitute $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$. (Source: IMO 1995, #2)
2. Let x,y,z be such that $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x}$. Then let $p=x+y-z, q=y+z-x, r=z+x-y$. Just remember that p,q,r are not necessarily positive! You can also multiply everything out to get Schur's inequality. (Source: IMO 2000, #2)
3. Let $a=\tan X, b=\tan Y, c=\tan Z$, and write $ab+bc+ca=\frac{(a+b+c)^2}{2}-\frac{a^2+b^2+c^2}{2}$. (Source: Iran 2006)

More Examples: Isolated Fudging

1. The denominators are messy. Simplify using the fact that $(a^2+b+c)(1+b+c) \geq (a+b+c)^2$. (Source: Adapted from Ukraine 2008).

2. By Muirhead, $a^3+b^3 \geq a^2b+ab^2$. Alternatively, once you think to use isolated fudging, you should always try comparing with simple stuff like $\frac{a}{a+b+c}$. (Source: USAMO 1997, #5)
3. Prove $\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq \frac{3a}{a+b+c}$. (Source: MOP 2002)
4. $1-xy = (x^2+y^2-xy)+(z^2)$. If you are clever, you can use Cauchy-Schwarz twice and never have to do anything messy. (Source: Pham Van Thuan and Le Vi)

Problems

A1. This is a quadratic equation in a. When is a quadratic equation minimized? It is also a sum of two squares.

A2. $44 \approx \sqrt{1999}$. Can you change some terms or add some terms to make the product telescope? (Source: CMO 1997, #3)

A3. Can you use the constraint to make the inequality homogenous? (This means that all terms have the same degree, so multiplying a,b,c by a constant does not change anything.) If you can do that, you can forget about the constraint and everything becomes much simpler! (Source: CMO 2008, #3)

A4. This problem is a trap. It looks set up for Hölder, but really you should just multiply it out. (Source: APMO 1998, #3)

A5. Apply Cauchy-Schwarz in Engel form. (Source: IMO 1982, #3)

A6. Do the Ravi substitution and expand. As a rule of thumb, if an inequality can be expanded in just a few minutes, you should always try it. (Source: IMO 1983, #6)

B1. Assume $a \geq b \geq c$ and then apply the rearrangement inequality. (Source: Samin Riasat)

B2. The condition can be rewritten as $\sum_{\text{cyc}} \frac{x-1}{x} = 1$. Now apply Cauchy-Schwarz. (Source: Iran 1998)

B3. Let $a = k \cdot \frac{x}{y}, b = k \cdot \frac{y}{z}, c = k \cdot \frac{z}{x}$. (Source: Balkan MO 2006 and other places)

B4. Use induction. (Source: APMO 1999, #2)

B5. One option is to use the fact $f(x) = \frac{1}{\sqrt{x}}$ is convex. Alternatively, you can apply Hölder's inequality on (left-hand side)$^2 \cdot$ something. (Source: Pham Kim Hung and Vo Quoc Ba Can)

B6. Prove the following algebraic identity: $\left(\sum_{i,j,k} a_{i,j}b_{j,k}c_{k,i}\right)^2 \leq \sum_{i,j} a_{i,j}^2 \sum_{j,k} b_{j,k}^2 \sum_{k,i} c_{k,i}^2$. You can use Cauchy-Schwarz. (Source: IMO 1992, #5)

B7. The minimum value is $\frac{3}{2}$. Try flipping the sign and simplifying a bit (section 3.1), then using induction. (Source: China Girls 2007)

C1. I haven't solved this yet! (Source: USAMO 2000, #6).

C2. I haven't solved this yet! (Source: IMO 2003 Shortlist, A6).